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Lattice Grids and Prisms are Antimagic

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Abstract

An *antimagic labeling* of a finite undirected simple graph with m edges and n vertices is a bijection from the set of edges to the integers $1, \ldots, m$ such that all n vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with the same vertex. A graph is called *antimagic* if it has an antimagic labeling. In 1990, Hartsfield and Ringel conjectured that every connected graph, but K_2, is antimagic. In 2004, N. Alon et al showed that this conjecture is true for n-vertex graphs with minimum degree $\Omega(\log n)$. They also proved that complete partite graphs (other than K_2) and n-vertex graphs with maximum degree at least $n-2$ are antimagic. Recently, Wang showed that the toroidal grids (the Cartesian products of two or more cycles) are antimagic. Two open problems left in Wang's paper are about the antimagicness of lattice grid graphs and prism graphs, which are the Cartesian products of two paths, and of a cycle and a path, respectively. In this article, we prove that these two classes of graphs are antimagic, by constructing such antimagic labelings.

Keywords: Antimagic; Labeling; Lattice grid; Prism

1 Introduction

All graphs in this paper are finite, undirected and simple. In 1990, Hartsfield and Ringel [3] introduced the concept of *antimagic* graph. An *antimagic labeling* of a graph with m edges and n vertices is a bijection from the set of edges to the integers $1, \ldots, m$ such that all n vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with that vertex. A graph is called antimagic if it has an antimagic labeling. Hartsfield and Ringel showed that paths $P_n (n \geq 3)$, cycles, wheels, and complete graphs $K_n (n \geq 3)$ are antimagic. They conjectured that all trees except K_2 are antimagic. Moreover, all connected graphs except K_2 are antimagic. These two conjectures are unsettled. In 2004, Alon et al [1] showed that the latter conjecture is true for all graphs with n vertices and minimum degree $\Omega(\log n)$. They also proved that a graph G with n (≥ 4) vertices and maximum degree $\Delta(G) \geq n-2$ is antimagic, and all complete partite graphs except K_2 are antimagic. In [5], Wang showed that the toroidal grids (the Cartesian products of two cycles) are antimagic, the author also proved that all Cartesian products of an antimagic k-regular graph $(k > 1)$ and a cycle (consequently Cartesian products of more than two cycles) are antimagic. Two open problems left in [5] are about the antimagicness of lattice grid graphs and prism graphs, which are the Cartesian products of two paths, and of a cycle and a path, respectively.

In this paper, we prove that these two classes of graphs are antimagic, by constructing such antimagic labelings. In contrast to toroidal grids, lattices and prisms have less symmetry (more local structures), we will incorporate new strategies in our labeling. Our main results are the following two theorems, which are proved in Section 3 and Section 4, respectively.

Theorem 1.1 *All lattice grid graphs $P_1[m+1] \times P_2[n+1]$ are antimagic, for integers $m, n \geq 1$.*

Theorem 1.2 *All prism graphs $C[m] \times P[n+1]$ are antimagic, for integers $m \geq 3, n \geq 1$.*

For more results, open problems and conjectures on antimagic graphs and various graph labeling problems, please see [2, 4].

2 Preliminaries

The *Cartesian product* $G_1 \times G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph with vertex set $V_1 \times V_2$, and (u_1, u_2) is adjacent to (v_1, v_2) in $G_1 \times G_2$ if and only if $u_1 = v_1$ and $u_2 v_2 \in E_2$, or, $u_2 = v_2$ and $u_1 v_1 \in E_1$. The Cartesian product of two paths is a lattice grid graph, and the Cartesian product of a path and a cycle is a prism grid graph.

Before proving our main results, we first describe antimagic labeling on paths and cycles respectively (see Figure 1). The labeling methods are the same as in [5], here we rephrase them for the sake of completeness.

Lemma 2.1 *All paths $P[m+1]$ are antimagic for integers $m \geq 2$.*

Proof: Suppose the vertex set is $\{v_1, \ldots, v_{m+1}\}$ and the edge set is arranged to be $\{v_i v_{i+2} | i = 1, \ldots, m-1\} \cup \{v_m v_{m+1}\}$. The following labeling $f(v_i v_{i+2}) = i$, for $1 \leq i \leq m-1$, and $f(v_m v_{m+1}) = m$ is antimagic, since we have

$$f^+(v_i) = \begin{cases} i & i = 1, 2; \\ 2i - 2 & i = 3, \ldots, m; \\ 2m - 1 & i = m + 1. \end{cases}$$

Therefore,

$$f^+(v_1) < f^+(v_2) < \ldots\ldots < f^+(v_{m+1})$$

■

Lemma 2.2 *All cycles $C[m]$ are antimagic for integers $m \geq 3$.*

Proof: Suppose the vertex set is $\{v_1, \ldots, v_m\}$ and the edge set is arranged to be $\{v_1 v_2\} \cup \{v_i v_{i+2} | i = 1, \ldots, m-2\} \cup \{v_{m-1} v_m\}$. The following labeling $f(v_1 v_2) = 1$, $f(v_i v_{i+2}) = i+1$, for $1 \leq i \leq m-2$, and $f(v_{m-1} v_m) = m$ is antimagic, since we have

$$f^+(v_i) = \begin{cases} 3 & i = 1; \\ 2i & i = 2, \ldots, m-1; \\ 2m - 1 & i = m. \end{cases}$$

Therefore,

$$f^+(v_1) < f^+(v_2) < \ldots\ldots < f^+(v_m)$$

■

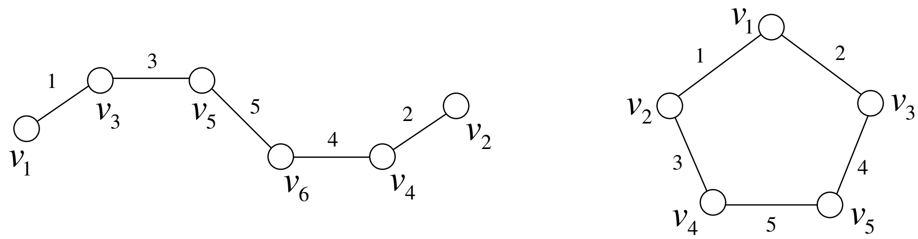

Fig. 1. Antimagic labeling of $P[n+1]$ and $C[m]$, for $n=5$, $m=5$

3 Proof of Theorem 1.1

Let $f : E(P_1[m+1]\times P_2[n+1]) \to \{1,2,\dots,2mn+m+n\}$ be an edge labeling of $P_1[m+1]\times P_2[n+1]$, and denote the induced sum at vertex (u,v) by $f^+(u,v)=\sum f((u,v),(y,z))$, where the sum runs over all vertices (y,z) adjacent to (u,v) in $P_1[m+1]\times P_2[n+1]$. To prove Theorem 1.1, first, we construct a labeling that is antimagic on product graphs of two paths $P_1[m+1]$ and $P_2[n+1]$, for $n \ge m \ge 2$. Then, we give an antimagic labeling of graphs $P_1[2]\times P_2[n+1]$, for $n \ge 1$.

3.1 $P_1[m+1]\times P_2[n+1]$ is Antimagic, for $n \ge m \ge 2$

Assume that $P_1[m+1]$ has edge set $\{u_iu_{i+2} | i = 1,\dots,m-1\} \cup \{u_mu_{m+1}\}$, and $P_2[n+1]$ has edge set $\{v_iv_{i+1} | i=1,\dots,n\}$. We will construct an antimagic labeling of $P_1[m+1]\times P_2[n+1]$ for $n \ge m \ge 2$, which contains two phases.

Phase 1: For the $mn+m$ edges contained in copies of $P_1[m+1]$ component (i.e., the edges $((u_i,v_j),(u_{i+2},v_j))$ and $((u_m,v_j),(u_{m+1},v_j))$, for $1 \le i \le m-1, 1 \le j \le n+1$), label them with even numbers $2,4,\dots,2mn+2m$ (notice $n \ge m$).

Specifically, first label the edges of $P_1[m+1]$ with U and R such that u_1u_3 is labeled with U, and two edges are labeled with different letters if they are incident to a same vertex. Obviously, there is one unique such labeling. For each edge $u_iu_j \in E(P_1[m+1])$ labeled with U, label the edges $((u_i,v_1),(u_j,v_1)),((u_i,v_2),(u_j,v_2)),\dots\dots,((u_i,v_{n+1}),(u_j,v_{n+1}))$ in usual order; for each edge $u_iu_j \in E(P_1[m+1])$ labeled with R, label the edges $((u_i,v_1),(u_j,v_1)),((u_i,v_2),(u_j,v_2)),\dots\dots,$ $((u_i,v_{n+1}),(u_j,v_{n+1}))$ in reversed order, and

$$\begin{array}{ll}
2,\ 4,\dots\dots\dots\dots\dots\dots\dots,2n+2, & \text{(labels for } ((u_1,v_i),(u_3,v_i)), i=1,2,\dots,n+1)\\
2n+4,\ 2n+6,\dots\dots\dots\dots,4n+4, & \text{(labels for } ((u_2,v_i),(u_4,v_i)), i=1,2,\dots,n+1)\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\
2mn+2m-2n,\dots,2mn+2m, & \text{(labels for } ((u_m,v_i),(u_{m+1},v_i)), i=1,2,\dots,n+1)
\end{array}$$

Phase 2: Denote by $A : a_1 < a_2 < \dots < a_s$ the sequence of all odd numbers in $\{1,2,\dots,2mn+m+n\}$, and denote by $B : b_1 < \dots < b_t$ the sequence of all even numbers in $\{2mn+2m+1,\dots,2mn+m+n\}$, i.e., the even numbers that are not used in Phase 1. Notice that $t \le \frac{1}{2}(2mn+m+n)-(mn+m) = \frac{1}{2}(n-m)$. We merge A and B into a sequence $C : a_1,a_2,\dots,a_{s-t},b_1,a_{s-t+1},b_2,\dots,b_t,a_s$ of $s+t$ terms ($s+t=mn+n$), and denote the sequence C by $c_1,c_2,...,c_{mn+n}$, which are the labels for the other $mn+n$ edges contained in copies of $P_2[n+1]$ component.

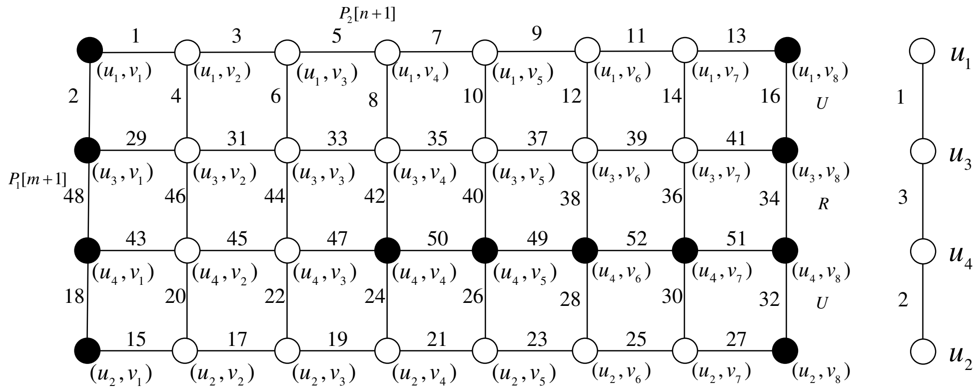

Fig. 2. Antimagic labeling of $P_1[m+1] \times P_2[n+1]$, for $m = 3, n = 7$

For the i-th $P_2[n+1]$ component (with vertices (u_i, v_1), (u_i, v_2),..., (u_i, v_{n+1})), label its edges in usual order according to the indices in the sequence C, $i = 1, 2, \ldots, m+1$, and

$$c_1,\ c_2, \ldots\ldots\ldots\ldots, c_n, \quad \text{(labels for the 1st } P_2[n+1] \text{ component)}$$

$$c_{n+1},\ c_{n+2}, \ldots\ldots\ldots, c_{2n}, \quad \text{(labels for the 2nd } P_2[n+1] \text{ component)}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$c_{mn+1}, c_{mn+2}, \ldots, c_{mn+n}, \quad \text{(labels for the } (m+1)\text{-th } P_2[n+1] \text{ component)}$$

Notice that $2t \leq n - m$, hence only the edges in the $(m+1)$-th $P_2[n+1]$ component may be labeled with even numbers (see Figure 2).

In what follows, we will show that the above labeling is antimagic. In the product graph $P_1[m+1] \times P_2[n+1]$, at each vertex (u, v), the edges incident to this vertex can be partitioned into two parts, one part is contained in a copy of $P_1[m+1]$ component, and the other part is contained in a copy of $P_2[n+1]$ component. Let $f_1^+(u, v)$ and $f_2^+(u, v)$ denote the sum at vertex (u, v) restricted to $P_1[m+1]$ component and $P_2[n+1]$ component respectively, i.e., $f_1^+(u, v) = \sum f((u, v), (y, v))$, where the sum runs over all vertices y adjacent to u in $P_1[m+1]$, and $f_2^+(u, v) = \sum f((u, v), (u, z))$, where the sum runs over all vertices z adjacent to v in $P_2[n+1]$. Therefore, $f^+(u, v) = f_1^+(u, v) + f_2^+(u, v)$. The following two claims imply the antimagicness of the above labeling.

Claim 3.1 *For the above labeling of $P_1[m+1] \times P_2[n+1]$, $n \geq m \geq 2$, we have*

$$\begin{aligned}
&f^+(u_1, v_2) < f^+(u_1, v_3) < \ldots\ldots\ldots < f^+(u_1, v_n) < \\
&f^+(u_2, v_2) < f^+(u_2, v_3) < \ldots\ldots\ldots < f^+(u_2, v_n) < \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&f^+(u_m, v_2) < f^+(u_m, v_3) < \ldots\ldots\ldots < f^+(u_m, v_n) < \\
&f^+(u_{m+1}, v_2) < \ldots < f^+(u_{m+1}, v_{n-2t}),
\end{aligned}$$

where t ($\leq \frac{1}{2}(n-m)$) is the number of even numbers in $\{2mn + 2m + 1, \ldots, 2mn + m + n\}$. In addition, all the above sums are even numbers.

Proof: Since $f_1^+(u_1,v_2) < f_1^+(u_1,v_3) < \ldots < f_1^+(u_1,v_n)$ and $f_2^+(u_1,v_2) < f_2^+(u_1,v_3) < \ldots < f_2^+(u_1,v_n)$, we have $f^+(u_1,v_2) < f^+(u_1,v_3) < \ldots < f^+(u_1,v_n)$. $f^+(u_1,v_n) < f^+(u_2,v_2)$ since $f_1^+(u_1,v_n) < f_1^+(u_2,v_2)$ and $f_2^+(u_1,v_n) < f_2^+(u_2,v_2)$. $f^+(u_2,v_2) < f^+(u_2,v_3) < \ldots\ldots < f^+(u_2,v_n)$ since $f_2^+(u_2,v_{i+1}) - f_2^+(u_2,v_i) \geq 4$ and $f_1^+(u_2,v_{i+1}) - f_1^+(u_2,v_i) \geq -2$, it follows that $f^+(u_2,v_{i+1}) - f^+(u_2,v_i) \geq 2$, for $i = 2,\ldots,n-1$. If $m = 2$, $f_1^+(u_3,v_2) = f_1^+(u_3,v_n) > f_1^+(u_2,v_n)$; if $m > 2$, $f_1^+(u_3,v_2) > f((u_3,v_2),(u_j,v_2)) > f((u_2,v_n),(u_4,v_n)) = f_1^+(u_2,v_n)$, where $j = 4$ or 5. Thus, in either case we have $f_1^+(u_2,v_n) < f_1^+(u_3,v_2)$. Clearly, $f_2^+(u_2,v_n) < f_2^+(u_3,v_2)$. It follows that $f^+(u_2,v_n) < f^+(u_3,v_2)$.

For the vertices of degree 4, clearly, $f_1^+(u_i,v_2) = f_1^+(u_i,v_3) = \ldots\ldots\ldots\ldots = f_1^+(u_i,v_n)$ for $i = 3,\ldots,m+1$. Moreover, $f_1^+(u_3,v_2) < f_1^+(u_4,v_2) < \ldots < f_1^+(u_{m+1},v_2)$ since $f((u_1,v_2),(u_3,v_2)) < f((u_2,v_2),(u_4,v_2)) < \ldots < f((u_{m-1},v_2),(u_{m+1},v_2)) < f((u_m,v_2),(u_{m+1},v_2))$. It follows that

$$\begin{array}{l}
f_1^+(u_3,v_2) = f_1^+(u_3,v_3) = \ldots\ldots\ldots\ldots = f_1^+(u_3,v_n) < \\
f_1^+(u_4,v_2) = f_1^+(u_4,v_3) = \ldots\ldots\ldots\ldots = f_1^+(u_4,v_n) < \\
\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
f_1^+(u_m,v_2) = f_1^+(u_m,v_3) = \ldots\ldots\ldots = f_1^+(u_m,v_n) < \\
f_1^+(u_{m+1},v_2) = \ldots = f_1^+(u_{m+1},v_{n-2t}).
\end{array}$$

On the other hand, since $c_1 < c_2 < \ldots < c_{mn+n-2t}$, we have that

$$\begin{array}{l}
f_2^+(u_3,v_2) < f_2^+(u_3,v_3) < \ldots\ldots\ldots\ldots < f_2^+(u_3,v_n) < \\
f_2^+(u_4,v_2) < f_2^+(u_4,v_3) < \ldots\ldots\ldots\ldots < f_2^+(u_4,v_n) < \\
\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
f_2^+(u_m,v_2) < f_2^+(u_m,v_3) < \ldots\ldots\ldots < f_2^+(u_m,v_n) < \\
f_2^+(u_{m+1},v_2) < \ldots < f_2^+(u_{m+1},v_{n-2t}).
\end{array}$$

Therefore,

$$\begin{array}{l}
f^+(u_3,v_2) < f^+(u_3,v_3) < \ldots\ldots\ldots\ldots < f^+(u_3,v_n) < \\
f^+(u_4,v_2) < f^+(u_4,v_3) < \ldots\ldots\ldots\ldots < f^+(u_4,v_n) < \\
\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
f^+(u_m,v_2) < f^+(u_m,v_3) < \ldots\ldots\ldots < f^+(u_m,v_n) < \\
f^+(u_{m+1},v_2) < \ldots < f^+(u_{m+1},v_{n-2t}).
\end{array}$$

All the above sums are even because each of them contains exactly two odd labels. ■

Claim 3.2 *The remaining* $2m+2+2t$ *sums* $f^+(u_1,v_1)$, $f^+(u_1,v_{n+1})$, $f^+(u_2,v_1)$, $f^+(u_2,v_{n+1}),\ldots,$ $f^+(u_{m+1},v_1)$, $f^+(u_{m+1},v_{n+1})$, *and* $f^+(u_{m+1},v_{n+1-2t})$, $f^+(u_{m+1},v_{n+2-2t}),\ldots,$ $f^+(u_{m+1},v_n)$ *are pairwise distinct. In addition, they are all odd numbers.*

Proof: Let us first consider the $2m+2$ sums $f^+(u_1,v_1)$, $f^+(u_1,v_{n+1})$, $f^+(u_2,v_1)$, $f^+(u_2,v_{n+1}),\ldots,$ $f^+(u_{m+1},v_1)$, $f^+(u_{m+1},v_{n+1})$, there are two natural cases:

Case 1. m is odd. In this case $u_2u_4 \in E(P_1[m+1])$ is labeled with U, from the way we do the

labeling, we have $f_1^+(u_1,v_1) \le f_1^+(u_1,v_{n+1}) \le f_1^+(u_2,v_1) \le f_1^+(u_2,v_{n+1}) \le \ldots \le f_1^+(u_{m+1},v_1) \le f_1^+(u_{m+1},v_{n+1})$ and $f_2^+(u_1,v_1) < f_2^+(u_1,v_{n+1}) < f_2^+(u_2,v_1) < f_2^+(u_2,v_{n+1}) < \ldots < f_2^+(u_{m+1},v_1) < f_2^+(u_{m+1},v_{n+1})$. Therefore, $f^+(u_1,v_1) < f^+(u_1,v_{n+1}) < f^+(u_2,v_1) < f^+(u_2,v_{n+1}) < \ldots < f^+(u_{m+1},v_1) < f^+(u_{m+1},v_{n+1})$.

Case 2. m is even. In this case $u_2u_j \in E(P_1[m+1])$ is labeled with R (where $j=3$ if $m=2$, $j=4$ if $m>2$), the ordering of the $2m+2$ sums $f^+(u_1,v_1)$, $f^+(u_1,v_{n+1})$, $f^+(u_2,v_1)$, $f^+(u_2,v_{n+1})$,..., $f^+(u_{m+1},v_1)$, $f^+(u_{m+1},v_{n+1})$ is the same as in case 1, but between vertices (u_2,v_1) and (u_2,v_{n+1}). Specifically, we have $f_1^+(u_1,v_1) \le f_1^+(u_1,v_{n+1}) \le f_1^+(u_2,v_1), f_1^+(u_2,v_{n+1}) \le f_1^+(u_3,v_1) \le \ldots \le f_1^+(u_{m+1},v_{n+1})$ and $f_2^+(u_1,v_1) < f_2^+(u_1,v_{n+1}) < f_2^+(u_2,v_1) < f_2^+(u_2,v_{n+1}) < \ldots < f_2^+(u_{m+1},v_1) < f_2^+(u_{m+1},v_{n+1})$. Therefore,

$$f^+(u_1,v_1) < f^+(u_1,v_{n+1}) < f^+(u_2,v_1), f^+(u_2,v_{n+1}) < \ldots < f^+(u_{m+1},v_1) < f^+(u_{m+1},v_{n+1}).$$

Since $f^+(u_2,v_1) = f_1^+(u_2,v_1) + f_2^+(u_2,v_1) = (4n+4)+(2n+1) = 6n+5$, and $f^+(u_2,v_{n+1}) = f_1^+(u_2,v_{n+1}) + f_2^+(u_2,v_{n+1}) = (2n+4)+(4n-1) = 6n+3$, it follows that $f^+(u_1,v_1) < f^+(u_1,v_{n+1}) < f^+(u_2,v_{n+1}) < f^+(u_2,v_1) < \ldots < f^+(u_{m+1},v_1) < f^+(u_{m+1},v_{n+1})$.

Thus, in any of the above two cases, the $2m+2$ sums $f^+(u_1,v_1)$, $f^+(u_1,v_{n+1})$, $f^+(u_2,v_1)$, $f^+(u_2,v_{n+1})$,..., $f^+(u_{m+1},v_1)$, $f^+(u_{m+1},v_{n+1})$ are pairwise distinct, and $f^+(u_{m+1},v_{n+1})$ is the largest among them. For the other $2t$ sums $f^+(u_{m+1},v_{n+1-2t})$, $f^+(u_{m+1},v_{n+2-2t})$,..., $f^+(u_{m+1},v_n)$, they are in strict increasing order $f^+(u_{m+1},v_{n+1-2t}) < f^+(u_{m+1},v_{n+2-2t}) < \ldots < f^+(u_{m+1},v_n)$, since $f_1^+(u_{m+1},v_{n+1-2t}) = f_1^+(u_{m+1},v_{n+2-2t}) = \ldots = f_1^+(u_{m+1},v_n)$ and $f_2^+(u_{m+1},v_{n+1-2t}) < f_2^+(u_{m+1},v_{n+2-2t}) < \ldots < f_2^+(u_{m+1},v_n)$.

At this point, the only remained issue is to notice that $f^+(u_{m+1},v_{n+1-2t}) > f^+(u_{m+1},v_{n+1})$, since $f_1^+(u_{m+1},v_{n+1-2t}) = f_1^+(u_{m+1},v_{n+1})$ and $f_2^+(u_{m+1},v_{n+1-2t}) = a_{s-t}+b_1 \ge (2mn+m+n-1-2t)+(2mn+2m+2) \ge 2mn+m+n-1-(n-m)+2mn+2m+2 = 4mn+4m+1 > 2mn+m+n \ge a_s = f_2^+(u_{m+1},v_{n+1})$. Hence, the $2m+2t+2$ sums are pairwise distinct. They are all odd numbers since each of them contains exactly one odd label. ■

Combining Claim 3.1 and Claim 3.2, we have proved that the above labeling of $P_1[m+1] \times P_2[n+1]$ is antimagic, for $n \ge m \ge 2$. Please see Figure 2 as an example of antimagic labeling of $P_1[m+1] \times P_2[n+1]$, for $m=3, n=7$.

3.2 $P_1[2] \times P_2[n+1]$ is Antimagic, for $n \ge 1$

Assume that $P_2[n+1]$ has edge set $\{v_iv_{i+2} | i = 1, \ldots, n-1\} \cup \{v_nv_{n+1}\}$. For $n=1$, $P_1[2] \times P_2[2]$ is isomorphic to $C[4]$, hence by Lemma 2.2, it is antimagic. For $n>1$, label $1,3,\ldots,2n-1$ to the edges $((u_1,v_1),(u_1,v_3))$, $((u_1,v_2),(u_1,v_4))$,......, $((u_1,v_{n-1}),(u_1,v_{n+1}))$,$((u_1,v_n),(u_1,v_{n+1}))$, label $2,4,\ldots,2n$ to the edges $((u_2,v_1),(u_2,v_3))$, $((u_2,v_2),(u_2,v_4))$,......, $((u_2,v_{n-1}),(u_2,v_{n+1}))$, $((u_2,v_n),(u_2,v_{n+1}))$, and label $2n+1, 2n+2, \ldots, 3n+1$ to $((u_1,v_1),(u_2,v_1))$, $((u_1,v_2),(u_2,v_2))$, ..., $((u_1,v_{n+1}),(u_2,v_{n+1}))$ (see Figure 3).

We will show that the above labeling (for $n>1$) is antimagic. Since the vertex sums restricted to $P_1[2]$ component satisfy that $f_1^+(u_1,v_1) = f_1^+(u_2,v_1) < f_1^+(u_1,v_2) = f_1^+(u_2,v_2) < \ldots < f_1^+(u_1,v_{n+1}) = f_1^+(u_2,v_{n+1})$ ('=' and '<' alternate), and the vertex sums restricted to $P_2[n+1]$ component are

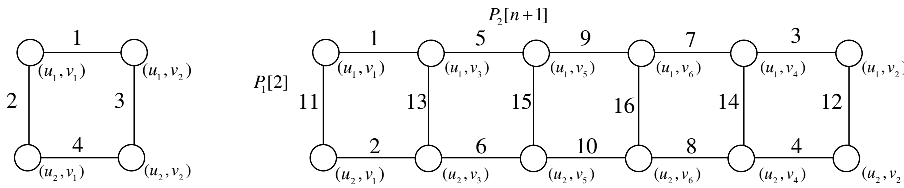

Fig. 3. Antimagic labelings of $P_1[2] \times P_2[2]$ and $P_1[2] \times P_2[n+1]$, for $n = 5$

$$f_2^+(u_1, v_i) = \begin{cases} 1 & i = 1; \\ 3 & i = 2; \\ 4i - 6 & i = 3, \ldots, n; \\ 4n - 4 & i = n + 1; \end{cases} \qquad f_2^+(u_2, v_i) = \begin{cases} 2 & i = 1; \\ 4 & i = 2; \\ 4i - 4 & i = 3, \ldots, n; \\ 4n - 2 & i = n + 1. \end{cases}$$

It follows that $f_2^+(u_1, v_1) < f_2^+(u_2, v_1) < f_2^+(u_1, v_2) < \ldots < f_2^+(u_2, v_n) = f_2^+(u_1, v_{n+1}) < f_2^+(u_2, v_{n+1})$ (there is one equality). Therefore, $f^+(u_1, v_1) < f^+(u_2, v_1) < f^+(u_1, v_2) < f^+(u_2, v_2) < \ldots < f^+(u_1, v_{n+1}) < f^+(u_2, v_{n+1})$, implying the antimagicness of the above labeling.

Combining the above two cases, we have proved Theorem 1.1.

4 Proof of Theorem 1.2

Assume that in the product graph $C[m] \times P[n+1]$, $C[m]$ has edge set $\{u_1u_2\} \cup \{u_iu_{i+2} | i = 1, \ldots, m-2\} \cup \{u_{m-1}u_m\}$, and $P[n+1]$ has edge set $\{v_iv_{i+2} | i = 1, \ldots, n-1\} \cup \{v_nv_{n+1}\}$. To prove Theorem 1.2, first, we construct a labeling that is antimagic on product graphs $C[m] \times P[n+1]$ for $m \geq 3, n \geq 2$. Then, we give an antimagic labeling of graphs $C[m] \times P[2]$ for $m \geq 3$.

Lemma 4.1 *$C[m] \times P[n+1]$ is antimagic for $m \geq 3, n \geq 2$.*

Proof: The antimagic labeling we will construct in this case ($m \geq 3, n \geq 2$) is similar with the labeling constructed in [5] on toroidal grids, the difference made here is to adapt the structure of prisms. The labeling contains two phases.

Phase 1: Using the same way as in the antimagic labeling of cycles in Lemma 2.2, label the edges on the i-th $C[m]$ component (with vertices (u_1, v_i), (u_2, v_i),..., (u_m, v_i)), for $i = 1, 2, \ldots, n+1$, and

$$1,\ 2, \ldots\ldots, m, \quad \text{(labels for the 1st } C[m] \text{ component)}$$

$$m+1,\ m+2, \ldots\ldots, 2m, \quad \text{(labels for the 2nd } C[m] \text{ component)}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$mn+1,\ mn+2, \ldots\ldots mn+m. \quad \text{(labels for the } (n+1)\text{-th } C[m] \text{ component)}$$

Phase 2: Similarly, label the edges of $P[n+1]$ with U and R such that v_1v_3 is labeled with U, and two edges are labeled with different letters if they are incident to a same vertex. For each edge $v_iv_j \in E(P[n+1])$ labeled with U, the edges $((u_1, v_i), (u_1, v_j))$, $((u_2, v_i), (u_2, v_j))$,......,$((u_m, v_i), (u_m, v_j))$ will be labeled in usual order; for each edge $v_iv_j \in E(P[n+1])$ labeled with R, the edges $((u_1, v_i), (u_1, v_j))$, $((u_2, v_i), (u_2, v_j))$,......,$((u_m, v_i), (u_m, v_j))$ will be labeled in reversed order, and

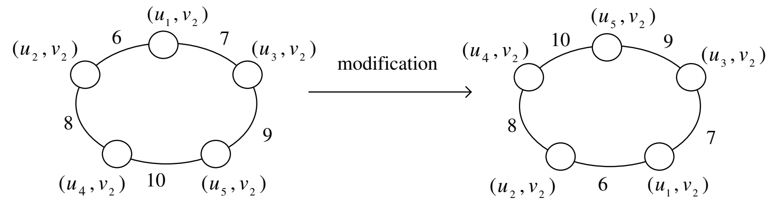

Fig. 4. Modification on the 2nd $C[m]$ component in case n is even, for $m = 5$

$$mn+m+1,\ mn+m+2, \dots, mn+2m, \quad \text{(labels for } ((u_i, v_1),(u_i, v_3)), i = 1, 2, \dots, m)$$
$$mn+2m+1, mn+2m+2, \dots, mn+3m, \quad \text{(labels for } ((u_i, v_2),(u_i, v_4)), i = 1, 2, \dots, m)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$2mn+1, 2mn+2, \dots\dots\dots, 2mn+m, \quad \text{(labels for } ((u_i, v_n),(u_i, v_{n+1})), i = 1, 2, \dots, m)$$

If $v_2v_j \in E(P[n+1])$ ($j = 3$ if $n = 2$, $j = 4$ if $n > 2$) is labeled with R (i.e., when n is even), we will take a *modification* process on the 2nd $C[m]$ component (with vertices (u_1, v_2), (u_2, v_2),..., (u_m, v_2)), which goes as follows. For each $u_iu_j \in E(C[m])$, the edge $((u_i, v_2),(u_j, v_2))$ will be relabeled with $(3m+1) - l_0(i, j)$, where $l_0(i, j)$ is the original label assigned to $((u_i, v_2),(u_j, v_2))$ in Phase 1 (i.e., we 'reverse' the labeling on the 2nd $C[m]$ component, whose edges will still be labeled with the same set of numbers $\{m+1, m+2, \dots, 2m\}$). Then, we rename each vertex (u_i, v_2) as (u_{m+1-i}, v_2), for $i = 1, 2, \dots, m$ (see Figure 4).

Let $f_1^+(u, v)$ and $f_2^+(u, v)$ be the vertex sum at $(u, v) \in V(C[m] \times P[n+1])$ restricted to $C[m]$ component and $P[n+1]$ component, respectively. Then, $f^+(u, v) = f_1^+(u, v) + f_2^+(u, v)$ is the vertex sum at (u, v). It is easy to see that, for the above labeling, independent of the parity of n (i.e., no matter whether there is a modification process or not), the ordering $f_1^+(u_1, v_2) < f_1^+(u_2, v_2) < \dots\dots < f_1^+(u_m, v_2)$ and $f_2^+(u_1, v_2) < f_2^+(u_2, v_2) < \dots\dots < f_2^+(u_m, v_2)$ will hold.

Using similar arguments, it is straightforward to prove that for the above labeling we have

$$\begin{gathered}
f_1^+(u_1, v_1) < f_1^+(u_2, v_1) < \dots\dots\dots\dots < f_1^+(u_m, v_1) < \\
f_1^+(u_1, v_2) < f_1^+(u_2, v_2) < \dots\dots\dots\dots < f_1^+(u_m, v_2) < \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
f_1^+(u_1, v_{n+1}) < f_1^+(u_2, v_{n+1}) < \dots\dots < f_1^+(u_m, v_{n+1}),
\end{gathered}$$

and

$$\begin{gathered}
f_2^+(u_1, v_1) \le f_2^+(u_2, v_1) \le \dots\dots\dots\dots \le f_2^+(u_m, v_1) \le \\
f_2^+(u_1, v_2) \le f_2^+(u_2, v_2) \le \dots\dots\dots\dots \le f_2^+(u_m, v_2) \le \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
f_2^+(u_1, v_{n+1}) \le f_2^+(u_2, v_{n+1}) \le \dots\dots \le f_2^+(u_m, v_{n+1}).
\end{gathered}$$

Therefore,

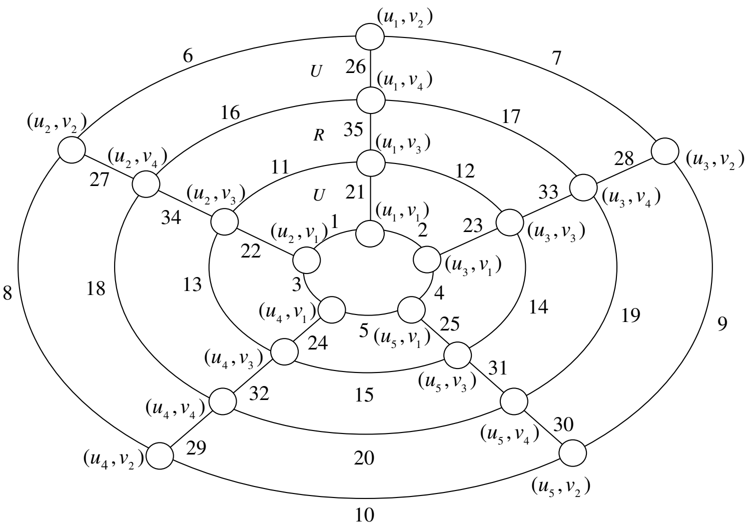

Fig. 5. Antimagic labeling of $C[m] \times P[n+1]$, for $m = 5$, $n = 3$

$$\begin{gathered}
f^+(u_1, v_1) < f^+(u_2, v_1) < \dots\dots\dots\dots < f^+(u_m, v_1) < \\
f^+(u_1, v_2) < f^+(u_2, v_2) < \dots\dots\dots\dots < f^+(u_m, v_2) < \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
f^+(u_1, v_{n+1}) < f^+(u_2, v_{n+1}) < \dots\dots < f^+(u_m, v_{n+1}),
\end{gathered}$$

which implies that the above labeling is antimagic. Please see Figure 5 as an example of antimagic labeling of $C[m] \times P[n+1]$, for $m = 5$, $n = 3$. ■

Lemma 4.2 *$C[m] \times P[2]$ is antimagic for $m \geq 3$.*

Proof: Assume that $C[m]$ has edge set $\{u_1u_2\} \cup \{u_iu_{i+2} | i = 1, \dots, m-2\} \cup \{u_{m-1}u_m\}$. Label $1, 3, \dots, 2m-1$ to the edges $((u_1, v_1), (u_2, v_1))$, $((u_1, v_1), (u_3, v_1))$,……, $((u_{m-2}, v_1), (u_m, v_1))$,$((u_{m-1}, v_1), (u_m, v_1))$, label $2, 4, \dots, 2m$ to the edges $((u_1, v_2), (u_2, v_2))$, $((u_1, v_2), (u_3, v_2))$,……, $((u_{m-2}, v_2), (u_m, v_2))$,$((u_{m-1}, v_2), (u_m, v_2))$, and label $2m+1, 2m+2, \dots, 3m$ to the edges $((u_1, v_1), (u_1, v_2))$, $((u_2, v_1), (u_2, v_2))$, ……, $((u_m, v_1), (u_m, v_2))$ (see Figure 6).

We will show that the above labeling ($m \geq 3$) is antimagic. Since the vertex sums restricted to $C[m]$ component are

$$f_1^+(u_i, v_1) = \begin{cases} 4 & i = 1; \\ 4i - 2 & i = 2, \dots, m-1; \\ 4m - 4 & i = m; \end{cases} \qquad f_1^+(u_i, v_2) = \begin{cases} 6 & i = 1; \\ 4i & i = 2, \dots, m-1; \\ 4m - 2 & i = m. \end{cases}$$

It follows that $f_1^+(u_1, v_1) < f_1^+(u_1, v_2) = f_1^+(u_2, v_1) < \dots < f_1^+(u_{m-1}, v_2) = f_1^+(u_m, v_1) < f_1^+(u_m, v_2)$ (there are two equalities). In addition, $f_2^+(u_1, v_1) = f_2^+(u_1, v_2) < f_2^+(u_2, v_1) = f_2^+(u_2, v_2) < \dots < f_2^+(u_m, v_1) = f_2^+(u_m, v_2)$ ('=' and '<' alternate). Therefore, $f^+(u_1, v_1) < f^+(u_1, v_2) <$

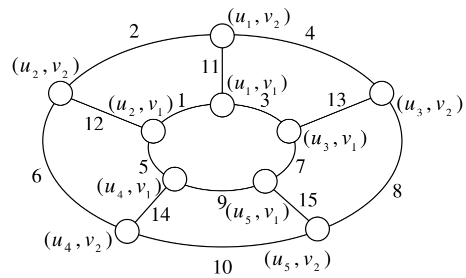

Fig. 6. Antimagic labeling of $C[m] \times P[2]$, for $m = 5$

$f^+(u_2, v_1) < f^+(u_2, v_2) < \ldots < f^+(u_m, v_1) < f^+(u_m, v_2)$, implying the antimagicness of the above labeling. ■

Combining Lemma 4.1 and Lemma 4.2, we have proved Theorem 1.2.

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